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Optimal Shape Design Problems for a Class of Systems Described by Hemivariational Inequalities

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Abstract. Optimal shape design problems for systems governed by an elliptic hemivariational inequality are considered. A general existence result for this problem is established by the mapping method.

Key words: Optimal shape design, hemivariational inequality, lower semicontinuity, Mosco convergence, Clarke subdifferential.

1. Introduction

The theory of variational inequalities (referred to as VIs) started in the sixties with papers of C. Baiocchi, H. Brezis, G. Duvaut, G. Fichera, D. Kinderlehrer, J. L. Lions, G. Stampacchia and others. It provides mathematical models for problems from elasticity, the fluid flow through porous media, semipermeable media etc. (see [11]). The theory of VIs is based on a variational method for PDEs and the calculus of variations, and it permits to characterize solutions of minimization problems for convex and differentiable functionals on closed convex sets. In this context, VIs can be regarded as a more general description of systems than the one based on PDEs. In the case of quadratic functionals, where the set on which the minimization is performed becomes a subspace of a vector space, VIs reduce to PDEs (the Euler equations).

The theory of VIs has been considerably enriched by the development in many directions, in which assumptions on differentiability and convexity of functionals have been dropped (see, e.g. Brezis [3], Moreau [19] and Panagiotopoulos [25] for the connections between VIs with convex potentials and the theory of monotone operators). Recently, Panagiotopoulos [24], [25], [26] has formulated inequality expressions called hemivariational inequalities (HVIs), which are derived with the help of the generalized gradient of Clarke [6]. The HVIs are generalizations of VIs and they cover boundary value problems for PDEs with nonmonotone, nonconvex and possibly multivalued laws (for the problems where the Clarke subdifferential is a pseudomonotone operator, we refer to [22]).

The aim of this paper is to prove an existence result for an optimal shape design problem for systems described by HVIs. It may be formulated as a control problem in which HVI appears as a state equation and the role of controls is played by sets from a family of admissible shapes. The cost functional to be minimized is of general integral form. We consider the shape optimization problem separately for two types of HVIs:

1^0) for HVI with a nonlinear discontinuous law in a domain Ω in $\mathbb{R}^N$: find $u \in K(\Omega)$ such that

$$a(u, v-u) + \int_\Omega j^0(u, v-u)\, dx \geq \langle f, v-u \rangle_{V \times V'}\ , \ \ \forall\, v \in K(\Omega) \qquad (1)$$

and

2^0) for HVI with a nonlinear law on the boundary $\partial\Omega$ of Ω: find $u \in K(\Omega)$ such that

$$a(u, v-u) + \int_{\partial\Omega} j^0(u, v-u)\, d\sigma \geq \langle f, v-u \rangle_{V \times V'}\ , \ \ \forall\, v \in K(\Omega). \qquad (2)$$

Above $a(\cdot,\cdot)$ is a bilinear form on $V = H^1(\Omega)$ or $V = H_0^1(\Omega)$, $K(\Omega)$ is a nonempty, closed, convex subset of V and j^0 denotes Clarke's directional derivative of a locally Lipschitz function $j: \mathbb{R} \to \mathbb{R}$ whose subdifferential ∂j describes the nonmonotone, nonconvex and possibly multivalued law, respectively, in Ω and on its boundary $\partial\Omega$. As it is known, in applications the set $K(\Omega)$ incorporates various unilateral conditions on Ω or on $\partial\Omega$. For examples of concrete situations which lead to problems (1) and (2), we refer to Section 6.

The proof of the existence of optimal shapes is based on the direct method of the calculus of variations. We also use the mapping method, introduced by Murat and Simon in [20] and [21], which provides both a class of admissible shapes and a topology in this class of domains. The admissible shapes are obtained as the images of a fixed open bounded subset of $\mathbb{R}^N$ by regular (see Section 2 for details) bijections in $\mathbb{R}^N$. The lower semicontinuity of objective functionals is considered with respect to the local convergence of functions (cf. Serrin [32]).

There is a rich literature on the mathematical theory of shape optimization problems. We only mention that optimal shape design problems for PDEs were considered by Murat and Simon [20], Pironneau [29], Sokolowski, Zolesio, Tartar, Allaire, Sverak, Masmoudi and many others, while VIs were considered by Haslinger, Neittaanmäki and Tiihonen [13], Tiihonen [33], Liu and Rubio [16] and Neittaanmäki [23]. For computational aspects of shape design problems, see Haslinger, Neittaanmäki [12], Salmenjoki [31], Miettinen, Mäkelä and Haslinger [18].

A slightly different approach to problems described by VIs with nonconvex potential was introduced by Degiovanni, Marino and Tosques [8] who used another notion of the subdifferential.

Another approach to shape optimization for systems described by PDEs is presented by Buttazzo and Dal Maso [4] who have used the Γ convergence theory. We also mention that the distributed-parameter optimal control problems for HVIs have been treated by Haslinger and Panagiotopoulos [14], [15]. For more papers on the subject we refer to the above mentioned works.

The plan of the paper is as follows. In Section 2 we recall the notation and preliminaries about the mapping method. Section 3 is devoted to the study of the HVI of the form (1). For such a problem we provide a result (see Proposition 1) on the closedness (in suitable topologies) of the graph of a mapping which, to every admissible shape, assigns the solution set of (1). It turns out that this property is crucial for our approach and it permits to get, in Section 4, an existence of optimal shapes for systems described by the HVI of the type (1). We also employ the Mosco convergence (as $m \to \infty$) of sets $T_m K(\Omega_m)$ obtained from $K(\Omega_m)$ through mappings T_m taken in a suitable class of transformations. In Section 5 we study a class of HVIs of the form (2) by analogous methods as those of Section 4. The optimal shape design result for this case is also given. In the final section, we conclude with remarks on some examples coming from mechanics to which our results can be applied.

The main result of Section 4 have been announced in its preliminary form in [9] and [10].

2. The Mapping Method

In this section we recall the notation and basic results on the mapping method which were established by Murat and Simon [20]. Roughly speaking, this method consists in finding the optimal shapes in a class of admissible domains gained as images of a fixed set. More precisely, we introduce the metric space of domains as follows.

Let C be a bounded open subset of $\mathbb{R}^N$ with a boundary ∂C of class $W^{i,\infty}$, $i \geq 1$ and such that $int\,\overline{C} = C$. Denoting by $\overline{C}(\mathbb{R}^N, \mathbb{R}^N)$ the space of uniformly continuous functions from $\mathbb{R}^N$ to $\mathbb{R}^N$, we consider the following spaces

$$W^{k,\infty}(\mathbb{R}^N, \mathbb{R}^N) = \left\{\varphi \mid D^\alpha\varphi \in L^\infty(\mathbb{R}^N, \mathbb{R}^N) \text{ for all } \alpha, 0 \leq |\alpha| \leq k\right\}$$

$$W^{k,\overline{c}}(\mathbb{R}^N, \mathbb{R}^N) = \{\varphi \mid D^\alpha\varphi \in L^\infty(\mathbb{R}^N, \mathbb{R}^N) \cap \overline{C}(\mathbb{R}^N, \mathbb{R}^N) \text{ for all } \alpha \text{ and } 0 \leq |\alpha| \leq k\}.$$

Let σ denote the index equal to ∞ or to $\overline{c}$ and let $k \geq 1$. We define the space $\mathcal{O}^{k,\sigma}$ of bounded open sets of $\mathbb{R}^N$ which are isomorphic with C, i.e.

$$\mathcal{O}^{k,\sigma} = \{\Omega \mid \Omega = T(C),\ T \in \mathcal{F}^{k,\sigma}\},$$

where $\mathcal{F}^{k,\sigma}$ is the space of regular bijections in $\mathbb{R}^N$ defined by

$$\mathcal{F}^{k,\sigma} = \{T\colon \mathbb{R}^N \to \mathbb{R}^N \mid T \text{ is bijective and } T, T^{-1} \in \mathcal{V}^{k,\sigma}\},$$

$$\mathcal{V}^{k,\sigma} = \{T\colon \mathbb{R}^N \to \mathbb{R}^N \mid T - I \in W^{k,\sigma}(\mathbb{R}^N, \mathbb{R}^N)\}.$$

In other words, $\mathcal{F}^{k,\infty}$ represents the set of essentially bounded perturbations (with essentially bounded derivatives) of identity in $\mathbb{R}^N$. It can be seen (see [20]) that if

C has a $W^{k,\sigma}$ boundary, where $k \geq 1$ and $\sigma = \infty$ or $\overline{c}$, then every set $\Omega \in \mathcal{O}^{k,\sigma}$ also has the boundary of class $W^{k,\sigma}$. Endowing the space $[W^{k,\sigma}(\mathbb{R}^N)]^N$ with the norm

$$||\varphi||_{k,\sigma} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^N} \left(\sum_{0 \leq |\alpha| \leq k} |D^\alpha \varphi|^2_{\mathbb{R}^N} \right)^{1/2},$$

we define on $\mathcal{O}^{k,\sigma} \times \mathcal{O}^{k,\sigma}$ a function

$$\delta_{k,\sigma}(\Omega_1, \Omega_2) = \inf_{T \in \mathcal{F}^{k,\sigma}, T(\Omega_1) = \Omega_2} \left(||T - I||_{k,\sigma} + ||T^{-1} - I||_{k,\sigma} \right).$$

The mapping $\delta_{k,\sigma}$ for $k \geq 1$ and $\sigma = \infty$ or $\overline{c}$ is a pseudo-distance on $\mathcal{O}^{k,\sigma}$ since it does not satisfy the triangle inequality (see Section 2.4 of [20] for a precise definition). From Proposition 2.3, Theorem 2.2 and Theorem 2.4 of [20], we have

THEOREM 1. *Let $k \geq 1$ and $\sigma = \infty$ or $\overline{c}$.*

(1) There exists a positive constant μ_k such that $d_{k,\sigma}$ defined by $d_{k,\sigma} = \left(\delta_{k,\sigma} \wedge \mu_k\right)^{1/2}$ is a metric on $\mathcal{O}^{k,\sigma}$.

(2) The space $\left(\mathcal{O}^{k,\sigma}, d_{k,\sigma}\right)$ is a complete metric space.

(3) If $k \geq 2$, then the injection from $\mathcal{O}^{k,\sigma}$ into $\mathcal{O}^{k-1,\sigma}$ is compact. More precisely, if $k \geq 2$ and $\mathcal{B}$ is a bounded (in $\delta_{k,\sigma}$), closed subset of $\mathcal{O}^{k,\sigma}$, then for any sequence $\{\Omega_m\} \subset \mathcal{B}$, there exist a subsequence $\{\Omega_{m\nu}\}$ of $\{\Omega_m\}$ and a set $\Omega \in \mathcal{B}$ such that $\Omega_{m\nu} \to \Omega$ in $\mathcal{O}^{k-1,\sigma}$.

REMARK 1. It is known (cf. Section 2 in [20]) that (for $k \geq 1$ and $\sigma = \infty$ or $\overline{c}$), $\Omega_m \to \Omega$ in $\mathcal{O}^{k,\sigma}$ iff there exist T_m and T in $\mathcal{F}^{k,\sigma}$ such that $T_m(C) = \Omega_m$, $T(C) = \Omega$ and $T_m \to T$, $T_m^{-1} \to T^{-1}$ in $[W^{k,\sigma}(\mathbb{R}^N)]^N$.

Some facts on the mapping method, needed in this paper, are summarized in the following.

LEMMA 1.

(a) If $T \in \mathcal{F}^{1,\infty}$, $\Omega = T(C)$, then $u \in L^2(\Omega)$ iff $u \circ T \in L^2(C)$; $u \in H^1(\Omega)$ iff $u \circ T \in H^1(C)$. Moreover, if $u_m \to u$ in $H^1(\Omega)$ (or in $H^1(C)$) and $T \in \mathcal{F}^{k,\infty}$ with $k \geq 1$, then $u_m \circ T \to u \circ T$ in $H^1(C)$ (or $u_m \circ T^{-1} \to u \circ T^{-1}$ in $H^1(\Omega)$).

(b) Let $u \in H^l(\mathbb{R}^N)$ with $l = 0$ or 1, $k \geq 1$ and $\sigma = \infty$ or $\overline{c}$. Then the mapping $T \mapsto u \circ T$ is continuous from $\mathcal{V}^{k,\sigma}$ to $H^l(\mathbb{R}^N)$ at every point $T \in \mathcal{F}^{k,\sigma}$.

(c) Let $k \geq 1$ and $\sigma = \infty$ or $\overline{c}$. The following mappings are continuous

$$T \mapsto J_T^{-1} \text{ from } \mathcal{V}^{k,\sigma} \text{ to } W^{k-1,\sigma}(\mathbb{R}^N, \mathbb{R}^{2N}),$$

$$T \mapsto \det J_T \text{ from } \mathcal{V}^{k,\sigma} \text{ to } W^{k-1,\sigma}(\mathbb{R}^N, \mathbb{R})$$

at every point $T \in \mathcal{F}^{k,\sigma}$ (J_T denotes here the standard Jacobian matrix of T).

For the proofs of (a)–(c) of the above lemma, we refer, respectively to Lemma 4.1 (see also [16]), Lemma 4.4 (i) and Lemma 4.3 and 4.2 of [20].

In what follows, we report on relationships between the convergence in $\mathcal{O}^{k,\infty}$ and other types of convergence of sets.

Let D be an open set of $\mathbb{R}^N$ and let 1_D denote its characteristic function.

DEFINITION 1. The Hausdorff complementary topology, denoted by H^c, is given by the metric

$$d(\Omega_1, \Omega_2) = \max\left(\sup_{x\in D\setminus\Omega_1} \inf_{y\in D\setminus\Omega_2} ||x-y||, \sup_{x\in D\setminus\Omega_2} \inf_{y\in D\setminus\Omega_1} ||x-y||\right).$$

REMARK 2. Let $k \geq 1$.
(i) If $\Omega_m \to \Omega_0$ in $\mathcal{O}^{k,\infty}$, then $1_{\Omega_m} \to 1_{\Omega_0}$ in $L^2(\mathbb{R}^N)$;
(ii) If $\Omega_m \to \Omega_0$ in $\mathcal{O}^{k,\infty}$ and $int\,\overline{C} = C$, then $\Omega_m \xrightarrow{H^c} \Omega_0$.

The following important property of the H^c convergence is the “covering” of the compacts.

REMARK 3. If $\Omega_m \xrightarrow{H^c} \Omega_0$, then

$$\forall\, G \subset\subset \Omega_0,\ \exists\, m_G \in \mathbb{N} \,:\, \forall\, m \geq m_G \ \ G \subseteq \Omega_m.$$

Finally, we note that the Hausdorff H^c distance is not sufficient for our considerations since $\mathcal{O}^{k,\sigma}$ (for $k \geq 1$, $\sigma = \infty$ or $\overline{c}$) endowed with H^c distance is not complete (see Section 2 of [20]).

The following basic hypothesis will be needed in the next sections:

$\underline{(H_0)}$: C is a bounded open set in $\mathbb{R}^N$ with boundary of class $W^{i,\sigma}$, $i \geq 1$ such that $int\,\overline{C} = C$ and $\mathcal{B}$ is a bounded closed subset of $\mathcal{O}^{k,\sigma}$, with $k \geq 2$ and $1 \leq i \leq k$.

In Sections 3 and 4 we suppose that (H_0) is satisfied with $\sigma = \infty$, while in Section 5 this hypothesis for $\sigma = \overline{c}$ is assumed.

3. Hemivariational Inequality with Nonlinear Law in Ω

In this section we investigate a class of hemivariational inequalities with nonlinear laws appearing in Ω. After introducing notations, we present a result on the existence of solutions. Then, we show a priori estimates for solutions as well as a result on the dependence of the solution set on the domain.

Let us fix a set Ω in $\mathcal{B}$ where $\mathcal{B}$ denotes a bounded closed subset of $\mathcal{O}^{k,\infty}$ with $k \geq 2$, let $V = V(\Omega) = H^1(\Omega)$ with the usual norm denoted by $||\cdot||$ and let

$K(\Omega)$ be a nonempty closed convex subset of V. Let $a: V \times V \to \mathbb{R}$ be a bilinear, continuous form

$$a(u,v) = \int_\Omega \left[(A(x)\nabla u, \nabla v) + a_0(x)uv \right] \, dx. \tag{3}$$

Given $\beta \in L^\infty_{loc}(\mathbb{R})$, we denote by $\widehat{\beta}: \mathbb{R} \to 2^\mathbb{R}$ a multifunction obtained from β by filling in the gaps at its discontinuity points, i.e.

$$\widehat{\beta}(\xi) = [\underline{\beta}(\xi), \overline{\beta}(\xi)],$$

where

$$\underline{\beta}(\xi) = \lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|t-\xi|\le\delta} \beta(t), \quad \overline{\beta}(\xi) = \lim_{\delta\to 0^+} \operatorname*{ess\,sup}_{|t-\xi|\le\delta} \beta(t)$$

and $[\cdot,\cdot]$ denotes the interval. It is well known (cf. Chang [5]) that a locally Lipschitz function $j: \mathbb{R} \to \mathbb{R}$ can be determinated up to an additive constant by the relation $j(\xi) = \int_0^\xi \beta(s)\, ds$ and that $\partial j(\xi) \subset \widehat{\beta}(\xi)$. If moreover $\beta(\xi \pm 0)$ exist for every $\xi \in \mathbb{R}$, then $\partial j(\xi) = \widehat{\beta}(\xi)$. Here $\partial j: \mathbb{R} \to 2^\mathbb{R}$ denotes the Clarke's generalized subdifferential of j (see [6]) given by

$$\partial j(\xi) = \{\eta \in \mathbb{R} \mid j^0(\xi;\gamma) \ge \eta\gamma, \ \forall\, \gamma \in \mathbb{R}\} \quad \text{for all } \xi \in \mathbb{R}.$$

The notation $j^0(\cdot;\cdot)$ stands for Clarke's directional derivative defined by

$$j^0(\xi;\gamma) = \limsup_{h\to 0,\ t\downarrow 0} \frac{j(\xi+h+t\gamma) - j(\xi+h)}{t} \quad \text{for all } \xi, \gamma \in \mathbb{R}.$$

By a hemivariational inequality we mean the following problem:

$$\begin{cases} \text{find } u \in K(\Omega) \text{ such that} \\ a(u, v-u) + \displaystyle\int_\Omega j^0(u, v-u)\, dx \ge \langle f, v-u\rangle_{V'\times V}, \ \forall\, v \in K(\Omega). \end{cases} \tag{HVI}$$

We will make the following hypotheses concerning the data of the problem (HVI).

$\underline{H(a)}$: $a: V\times V \to \mathbb{R}$ is a bilinear, continuous (i.e. $|a(u,v)| \le M||u||\,||v||$ for $u, v \in V$ with $M > 0$) and symmetric form given by (3) which is coercive on V (i.e. $a(v,v) \ge \alpha||v||^2$ for $v \in V$) with $\alpha > 0$ independent of Ω and the matrix $A \in [C(\mathbb{R}^N)]^{N^2} \cap [L^\infty(\mathbb{R}^N)]^{N^2}$, $a_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $a_0(x) \ge \widetilde{a} > 0$ a.e. $x \in \Omega$.

$\underline{H(\beta)}$: $\beta \in L^\infty_{loc}(\mathbb{R})$ is such that

(i) $\beta(\xi \pm 0)$ exists for each $\xi \in \mathbb{R}$;

(ii) the graph of β ultimately increases i.e. there exist $\overline{\xi} \in \mathbb{R}$ such that

$$\operatorname*{ess\,sup}_{(-\infty,-\overline{\xi})} \beta(\xi) \leq 0 \leq \operatorname*{ess\,inf}_{(\overline{\xi},+\infty)} \beta(\xi);$$

(iii) there exists $c_0 > 0$ such that $|\beta(\xi)| \leq c_0(1+|\xi|)$ for $\xi \in \mathbb{R}$.

$\underline{H(K)_1}$: $K(\Omega)$ is a nonempty convex, closed subset of $V(\Omega)$,

$\underline{H(f)}$: $f \in V'$

The concept of solution to problem (HVI) is specified below.

DEFINITION 2. An element $u \in K(\Omega)$ is said to be a solution to (HVI) if there exists $\chi \in L^2(\Omega)$ such that

$$a(u, v-u) + \int_\Omega \chi(v-u)\, dx \geq \langle f, v-u \rangle_{V \times V'}\ ,\ \ \forall\, v \in K(\Omega) \tag{4}$$

and

$$\chi(x) \in \partial j(u(x)) \text{ for a.e. } x \in \Omega. \tag{5}$$

For a justification of this definition we refer to Chapter 3 of Naniewicz and Panagiotopoulos [22]. In the sequel, by $S(\Omega)$ we denote the set of all solutions to (HVI).

The following existence result can be proved by the methods of Chapter 3.4 of Naniewicz and Panagiotopoulos, loc. cit. (compare also [30]).

THEOREM 2. *If hypotheses $H(a)$, $H(\beta)$, $H(K)_1$ and $H(f)$ hold, then problem (HVI) admits a solution, i.e. $S(\Omega) \neq \emptyset$.*

Due to the lack of convexity of j (or some additional growth condition on the function β, see Miettinen [17]), no uniqueness result for (HVI) can be obtained, so $S(\Omega)$ contains, in general, more than one element.

LEMMA 2. *Under the hypotheses of Theorem 2, if $u \in S(\Omega)$, then the following estimate holds:*

$$||u|| \leq C(v)\,(1 + \textit{meas}(\Omega) + ||f||_{V'})\ , \tag{6}$$

where $C(v)$ satisfies $0 < C(v) \leq b_1||v|| + b_2$ with b_1, b_2 depending on M, α and c_0, and v is an arbitrary element of $K(\Omega)$.

Proof. Let $u \in S(\Omega)$. This means that $u \in K(\Omega)$ and there is a function $\chi \in L^2(\Omega)$ such that (4) and (5) are satisfied. Since β ultimately increases (see $H(\beta)(ii)$), two positive real numbers ρ_1 and ρ_2 can be determined such that

$$\beta(\xi) \leq 0, \quad \text{if } \xi \leq -\rho_1;$$

$$\beta(\xi) \geq 0, \quad \text{if } \xi \geq \rho_1;$$

$$|\beta(\xi)| \leq \rho_2, \quad \text{if } |\xi| \leq \rho_1.$$

Hence, using the definition of $\widehat{\beta}$, we readily get

$$\begin{aligned}(\chi, u)_{L^2(\Omega)} &= \int_{|u(x)|\leq\rho_1} \chi u \, dx + \int_{|u(x)|\geq\rho_1} \chi u \, dx \geq \\ &\geq \int_{|u(x)|\leq\rho_1} \chi u \, dx \geq -\rho_1\rho_2 \, \text{meas}(\Omega).\end{aligned} \tag{7}$$

On the other hand, owing to $H(\beta)(iii)$, we have

$$|\chi| \leq c_2 \left(\sqrt{\text{meas}(\Omega)} + |u| \right), \tag{8}$$

where $c_2 > 0$ depends only on c_0 and $| \cdot |$ denotes the $L^2(\Omega)$-norm. From the coercivity of $a(\cdot, \cdot)$, (4) and (7), we obtain

$$\begin{aligned}\alpha||u - v||^2 &\leq a(v-u, v-u) \leq (\chi, v-u)_{L^2} + a(v, v-u) - \langle f, v-u \rangle \leq \\ &\leq (\chi, v)_{L^2} + \rho_1\rho_2 \, \text{meas}(\Omega) + M||v|| \, ||v-u|| + ||f||_{V'}||u-v||\end{aligned}$$

for every $v \in K(\Omega)$. By taking (8) into account and using the inequality $|\cdot| \leq c_1||\cdot||$ with $c_1 > 0$, we get

$$\begin{aligned}\alpha||u - v||^2 &\leq \rho_1\rho_2 \, \text{meas}(\Omega) + M||v|| \, ||v-u|| + ||f||_{V'}||u-v|| + \\ &+ c_2 \left(\sqrt{\text{meas}(\Omega)} + c_1||u|| \right) |v|\end{aligned}$$

for all $v \in K(\Omega)$, and subsequently

$$\begin{aligned}\alpha||u - v||^2 &\leq (M||v|| + ||f||_{V'} + c_1c_2|v|) \, ||u-v|| + \\ &+ \rho_1\rho_2 \, \text{meas}(\Omega) + c_2|v| \left(\sqrt{\text{meas}(\Omega)} + c_1||v|| \right).\end{aligned} \tag{9}$$

Hence

$$||u|| \leq C(v) \left(1 + \text{meas}(\Omega) + ||f||_{V'}\right),$$

where $C(v) \leq b_1||v|| + b_2$ and so the inequality (6) follows easily. □

The next result will be crucial in the proof of the main theorem. We need the following hypothesis:

$\underline{H(K)_2}$: $K = TK(\Omega) \subset H^1(C)$ is independent of T for all $T \in \mathcal{F}^{k,\infty}$, with $k \geq 2$.

PROPOSITION 1. *Let us assume that* (H_0) *with* $\sigma = \infty$, $H(a)$, $H(\beta)$, $H(K)_1$, $H(K)_2$ *hold and* $f \in L^2(\Omega)$. *Then the map* $\mathcal{B} \ni \Omega \mapsto S(\Omega) \subset V(\Omega)$ *has a closed graph in the following sense: if* $\Omega_m, \Omega_0 \in \mathcal{B}$, $\Omega_m \to \Omega_0$ *in* $\mathcal{O}^{k,\infty}$, $(k \geq 1)$, $u_m \in S(\Omega_m)$, $\widehat{u}_m = u_m \circ T_m$, $\widehat{u}_m \to u^*$ *weakly in* $H^1(C)$, *then* $u^* = u_0 \circ T_0$ *for some* $u_0 \in S(\Omega_0)$, *where* $\Omega_m = T_m(C)$ *and* $\Omega_0 = T_0(C)$.

Proof. We follow some ideas of Liu and Rubio [16]. Let $\Omega_m, \Omega_0 \in \mathcal{B}$ be such that $\Omega_m \to \Omega_0$ in $\mathcal{O}^{k,\infty}$ with $k \geq 1$, where $\Omega_m = T_m(C)$ and $\Omega_0 = T_0(C)$. By definition (cf. Remark 1) $T_m, T_0 \in \mathcal{F}^{k,\infty}$ and $T_m - T_0 \to 0$, $T_m^{-1} - T_0^{-1} \to 0$ in $[W^{k,\infty}(\mathbb{R}^N)]^N$. Without loss of generality, we suppose that $\det J_{T_m} > 0$ and $\det J_{T_0} > 0$ on $\mathbb{R}^N$. Let $u_m \in S(\Omega_m)$, i.e. $u_m \in K(\Omega_m)$ and there exists $\chi_m \in L^2(\Omega_m)$ such that

$$a(u_m, v - u_m) + \int_{\Omega_m} \chi_m(v - u_m)\, dx \geq (f, v - u_m)_{L^2}, \quad \forall\, v \in K(\Omega_m) \tag{10}$$

and

$$\chi_m(x) \in \partial j(u_m(x)) \text{ for a.e. } x \in \Omega_m. \tag{11}$$

By using the transformation $x = T_m(X)$, we rewrite (10), (11) as the following equivalent problem on the set C:

$$a_{T_m}(\widehat{u}_m, v - \widehat{u}_m) + (\widehat{\chi}_m, v - \widehat{u}_m) \geq (\widehat{f}_m, v - \widehat{u}_m), \quad \forall\, v \in K \subset H^1(C) \tag{12}$$

and

$$\widehat{\chi}_m(X) \in \partial j(\widehat{u}_m(X)) \text{ for a.e. } X \in C, \tag{13}$$

where $\widehat{u}_m = u_m \circ T_m$, $\widehat{A}_{T_m} = A \circ T_m$, $\widehat{a}_m = a \circ T_m$, $\widehat{\chi}_m = \chi_m \circ T_m$, $\widehat{f}_m = f \circ T_m$, and

$$\begin{aligned} a_{T_m}(\widehat{u}_m, v) \;=\; & \int_C \Big[\Big(J_{T_m}^{-1}(X) \widehat{A}_{T_m}(X) J_{T_m}^{-t}(X) \nabla \widehat{u}_m(X), \nabla v(X) \Big) + \\ & + \widehat{a}_m(X) \widehat{u}_m(X) v(X) \Big] \det J_{T_m}(X)\, dX, \end{aligned}$$

$$(\widehat{\chi}_m, v) = \int_C \widehat{\chi}_m(X) v(X) \det J_{T_m}(X)\, dX,$$

$$(\widehat{f}_m, v) = \int_C \widehat{f}_m(X) v(X) \det J_{T_m}(X)\, dX.$$

Since K satisfies $H(K)_2$, we may consider v in (12) to be fixed. Moreover, we know (see Lemma 1(a)) that $\widehat{u}_m \in H^1(C)$ and $\widehat{\chi}_m, \widehat{f}_m \in L^2(C)$.

Our goal is to pass to the limit, as $m \to +\infty$, in the problem (12), (13). By hypothesis

$$\widehat{u}_m \to u^* \quad \text{weakly in } H^1(C) \tag{14}$$

and since ∂C is regular enough to ensure the compactness of the embedding $H^1(C) \subset L^2(C)$, we have

$$\widehat{u}_m \to u^* \quad \text{in } L^2(C). \tag{15}$$

The set K is weakly closed (being closed and convex), so from (14) we obtain $u^* \in K$.

On the other hand, by using $H(\beta)(iii)$, from (13) we get $|\widehat{\chi}_m| \leq c_3(\text{meas}(C) + |\widehat{u}_m|)$ with $c_3 > 0$ ($|\cdot|$ being the norm in $L^2(C)$). Hence and from (15), after passing to a subsequence if necessary, we have

$$\widehat{\chi}_m \to \chi^* \quad \text{weakly in } L^2(C) \tag{16}$$

with $\chi^* \in L^2(C)$.

By Lemma 1(b), we know that $\widehat{f}_m \to \widehat{f}_0$ in $L^2(\mathbb{R}^N)$ with $\widehat{f}_0 = f_0 \circ T_0$. It can be verified that

$$(\widehat{f}_m, v - \widehat{u}_m) \to (\widehat{f}_0, v - u^*). \tag{17}$$

Indeed, we have

$$\begin{aligned}
&|(\widehat{f}_m, v - \widehat{u}_m) - -(\widehat{f}_0, v - u^*)| = \\
&\quad = \left| \int_C \widehat{f}_m(v - \widehat{u}_m) \, \text{det} J_{T_m} \, dX - \int_C \widehat{f}_0(v - u^*) \, \text{det} J_{T_0} \, dX \right| \leq \\
&\quad \leq ||\text{det} J_{T_m} - \text{det} J_{T_0}|| \int_C \left| \widehat{f}_m(v - \widehat{u}_m) \right| \, dX + \\
&\qquad + \left| \int_C \left[\widehat{f}_m(v - \widehat{u}_m) - \widehat{f}_0(v - u^*) \right] \, \text{det} J_{T_0} \, dX \right| .
\end{aligned}$$

The first term on the right hand side converges to zero since the sequences $\{\widehat{f}_m\}$, $\{\widehat{u}_m\}$ are bounded in $L^2(C)$ and $\text{det} J_{T_m} \to \text{det} J_{T_0}$ in $L^\infty(\mathbb{R}^N)$ (as a consequence of Lemma 1(c)). The second term on the right hand side also tends to zero due to (15) and the strong convergence of $\widehat{f}_m$ to $\widehat{f}_0$ in $L^2(C)$.

In an analogous way as we proved (17), we can show, using Lemma 1(c), (15) and (16) that

$$(\widehat{\chi}_m, v - \widehat{u}_m) \to (\chi^*, v - u^*). \tag{18}$$

Subsequently, from the assumptions on the matrix A, we deduce that $A(\cdot)$ is uniformly continuous on every bounded subset of $\mathbb{R}^N$. Since $T_m \to T_0$, $T_m^{-1} \to T_0^{-1}$ in $[W^{k,\infty}(C)]^N$ and $T_m(C)$, $T_0(C)$ are in a bounded set of $\mathbb{R}^N$, we obtain

$$\widehat{A}_{T_m} \to \widehat{A}_{T_0} \quad \text{in } [L^\infty(C)]^{N^2}.$$

Hence, from Lemma 1(c), we have

$$J_{T_m}^{-1}\widehat{A}_{T_m}J_{T_m}^{-t} \to J_{T_0}^{-1}\widehat{A}_{T_0}J_{T_0}^{-t} \quad \text{in } [L^\infty(C)]^{N^2}. \tag{19}$$

Let $v \in K$ be fixed. From the following inequality

$$\begin{aligned}
&|a_{T_m}(\widehat{u}_m, v-\widehat{u}_m) - a_{T_0}(\widehat{u}_m, v-\widehat{u}_m)| \le \\
&\le \left|\int_C \left(\left[J_{T_m}^{-1}\widehat{A}_{T_m}J_{T_m}^{-t} - J_{T_0}^{-1}\widehat{A}_{T_0}J_{T_0}^{-t}\right]\nabla\widehat{u}_m, \nabla v - \nabla\widehat{u}_m\right) \det J_{T_m}\, dX\right| + \\
&\quad + \left|\int_C \left(J_{T_0}^{-1}\widehat{A}_{T_0}J_{T_0}^{-t}\nabla\widehat{u}_m, \nabla v - \nabla\widehat{u}_m\right) [\det J_{T_m} - \det J_{T_0}]\, dX\right| + \\
&\quad + \left|\int_C \widehat{a}_m\widehat{u}_m(v-\widehat{u}_m)\,[\det J_{T_m} - \det J_{T_0}]\, dX\right| + \\
&\quad + \left|\int_C (\widehat{a}_m - \widehat{a}_0)\widehat{u}_m(v-\widehat{u}_m)\,\det J_{T_0}\, dX\right| \le \\
&\le ||\det J_{T_m}||\, ||J_{T_m}^{-1}\widehat{A}_{T_m}J_{T_m}^{-t} - J_{T_0}^{-1}\widehat{A}_{T_0}J_{T_0}^{-t}||\, ||\widehat{u}_m||_{H^1(C)}\, ||\widehat{u}_m - v||_{H^1(C)} + \\
&\quad + ||J_{T_0}^{-1}\widehat{A}_{T_0}J_{T_0}^{-t}||\, ||\det J_{T_m} - \det J_{T_0}||\, ||\widehat{u}_m||_{H^1(C)}\, ||\widehat{u}_m - v||_{H^1(C)} + \\
&\quad + ||\widehat{a}_m||\, ||\widehat{u}_m||\, ||v-\widehat{u}_m||\, ||\det J_{T_m} - \det J_{T_0}|| + \\
&\quad + ||\widehat{a}_m - \widehat{a}_0||\, ||\widehat{u}_m||\, ||v-\widehat{u}_m||\, ||\det J_{T_0}||,
\end{aligned}$$

by taking (14), (19) and Lemma 1(c) into account, we get

$$a_{T_m}(\widehat{u}_m, v-\widehat{u}_m) - a_{T_0}(\widehat{u}_m, v-\widehat{u}_m) \to 0. \tag{20}$$

Next, we will show that passing to the limit in (12) and (13), we obtain

$$a_{T_0}(u^*, v-u^*) + (\chi^*, v-u^*) \ge (\widehat{f}_0, v-u^*), \quad \forall\, v \in K, \tag{21}$$

$$\chi^*(X) \in \partial j(u^*(X)) \text{ for a.e. } X \in C. \tag{22}$$

In order to prove (21), it is enough to observe that using the weak-V lower semi-continuity of the function $v \mapsto a(v,v)$ and convergences (14), (17), (18) and (20), we get

$$\begin{aligned}
&(\widehat{f}_0, v-u^*) = \liminf_m (\widehat{f}_m, v-\widehat{u}_m) \le \\
&\le \liminf_m \left[a_{T_m}(\widehat{u}_m, v-\widehat{u}_m) + (\widehat{\chi}_m, v-\widehat{u}_m)\right] \le \\
&\le \limsup_m a_{T_0}(\widehat{u}_m, v-\widehat{u}_m) + \lim_m \left[a_{T_m}(\widehat{u}_m, v-\widehat{u}_m) - a_{T_0}(\widehat{u}_m, v-\widehat{u}_m)\right] + \\
&\quad + \lim_m (\widehat{\chi}_m, v-\widehat{u}_m) \le a_{T_0}(u^*, v-u^*) + (\chi^*, v-u^*).
\end{aligned}$$

Then, in order to obtain the limit relation for (13), we apply the convergence theorem (see [1], p. 273). By passing to subsequences, if necessary, from (15) and (16), we have

$$\widehat{u}_m \to u^* \quad \text{a.e. in } C,$$

$$\widehat{\chi}_m \to \chi^* \quad \text{weakly in } L^1(C).$$

Since the multifunction $\partial j(\cdot)$ is u.s.c. with nonempty, convex and compact values (see [6]), by exploiting the above convergences, we deduce from (13) by applying the convergence theorem that (22) holds.

Now we write down the problem (21), (22) in an equivalent form by employing the transformation $X = T_0^{-1}(x)$. To this end, we introduce functions $u_0 = u^* \circ T_0^{-1}$ and $\chi_0 = \chi^* \circ T_0^{-1}$. From the relations $J_{T_0^{-1}}(x) = J_{T_0}^{-1}(T_0^{-1}(x))$ and $\det J_{T_0}(T_0^{-1}(x)) \cdot \det J_{T_0^{-1}}(x) = 1$ a.e. on $\mathbb{R}^N$ (cf. respectively, Corollary 2.1 and page IV-7 of [20]), we have

$$\begin{cases} a(u_0, v - u_0) + (\chi_0, v - u_0)_{L^2(\Omega_0)} \geq (f, v - u_0)_{L^2(\Omega_0)}, \quad \forall\, v \in K \\ \chi_0(x) \in \partial j(u_0(x)) \text{ for a.e. } x \in \Omega_0. \end{cases}$$

Since $u^* \in K$ and $\chi_0 \in L^2(\Omega_0)$, we conclude that $u_0 \in S(\Omega_0)$ and $u^* = u_0 \circ T_0$. This completes the proof of the proposition. □

REMARK 4. With a few modifications in the proof above we can establish the validity of Proposition 3.1 for the case when f is taken from $H^{-1}(\Omega)$ and not from $L^2(\Omega)$. On the other hand, we can also carry out this proof when $H^1(\Omega)$ is replaced by $H_0^1(\Omega)$.

4. A Shape Optimization Problem

In this section we present the main result of this paper on existence of optimal shapes for systems governed by hemivariational inequalities.

The optimal shape design problem consists in solving the following control problem:

$$\begin{cases} \text{find } (\Omega^*, u^*) \in \bigcup\limits_{\Omega \in \mathcal{B}} (\Omega \times S(\Omega)) \text{ such that} \\ \mathcal{J}(\Omega^*, u^*) = \min\limits_{\Omega \in \mathcal{B}} \min\limits_{v \in S(\Omega)} \mathcal{J}(\Omega, v), \end{cases} \tag{23}$$

in which the control is the set Ω changing in the family $\mathcal{B} \subset \mathcal{O}^{k,\infty}$ ($k \geq 2$) of admissible shapes, $S(\Omega)$ denotes the solution set of hemivariational inequality (HVI) and the cost functional is of the integral form

$$\mathcal{J}(\Omega, u) = \int_\Omega L(x, u(x), \nabla u(x))\, dx. \tag{24}$$

In the case $\beta \equiv 0$, the set $S(\Omega)$ reduces under our hypotheses to one element and hemivariational inequality becomes variational inequality considered by Liu and Rubio [16].

For expositional convenience we give first an existence result for (23) under the assumption that $TK(\Omega)$ is independent of T (i.e. $H(K)_2$ holds) and then in the case when $TK(\Omega)$ varies with T. We recall the following

DEFINITION 3. (see Serrin [32]) Let D be an open subset of $\mathbb{R}^N$ and let ϕ_m, ϕ be locally summable functions defined in $\mathbb{R}^N$. We say that ϕ_m converges locally to ϕ in D if for any compact subset G of D, we have: ϕ_m is defined and summable on G, at least for m sufficiently large and $||\phi_m - \phi||_{L^1(G)} \to 0$.

We need also the additional hypothesis:

$\underline{H(J)}$: J is l.s.c. with respect to the local convergence in $\mathbb{R}^N$.

THEOREM 3. *If hypotheses* (H_0) *with* $\sigma = \infty$, $H(a)$, $H(\beta)$, $H(K)_1$, $H(K)_2$, $H(J)$ *hold and* $f \in L^2(\Omega)$, *then problem (23) admits at least one solution.*

Proof. We apply the direct method of the calculus of variations. Let $(\Omega_m, u_m) \in \bigcup_{\Omega \in \mathcal{B}} (\Omega \times S(\Omega))$ be a minimizing sequence for (23). Since $\mathcal{B}$ is compact in $\mathcal{O}^{k-1,\infty}$ (cf. Theorem 1(3)), we infer that there is a subsequence of Ω_m (still indexed by m) and a set $\Omega_0 \in \mathcal{B}$ such that $\Omega_m \to \Omega_0$ in $\mathcal{O}^{k-1,\infty}$. This means that there exist T_m, $T_0 \in \mathcal{F}^{k-1,\infty}$ such that $\Omega_m = T_m(C)$, $\Omega_0 = T_0(C)$ and $T_m - T_0 \to 0$, $T_m^{-1} - T_0^{-1} \to 0$ in $[W^{k-1,\infty}(\mathbb{R}^N)]^N$.

On the other hand, since $u_m \in S(\Omega_m)$, we obtain (6) for an arbitrary $v \in K(\Omega_m)$. Since $H(K)_2$ holds, we may consider v in (6) to be fixed in K and therefore we have the estimate

$$||u_m||_{V(\Omega_m)} \leq C \left(1 + \text{meas}(\Omega_m) + ||f||_{L^2(\Omega_m)}\right), \tag{25}$$

where $C = C(v) > 0$ is independent of m. From Remark 2(i), we have $1_{\Omega_m} \to 1_{\Omega_0}$ in $L^2(\mathbb{R}^N)$ which gives, in particular, that meas$(\Omega_m) \leq c_4$ and $||f||_{L^2(\Omega_m)} \leq c_5$ with $c_4, c_5 > 0$ independent of m. Therefore, (25) implies that $\{||u_m||_{V(\Omega_m)}\}$ lies in a bounded set in $\mathbb{R}$. Putting $\widehat{u}_m = u_m \circ T_m$ and using the inequalities in Remark 4.1 of [20] (see also Section 2 of [16]), we obtain that $\{\widehat{u}_m\}$ remains in a bounded set of $H^1(C)$. Thus, after passing to a next subsequence if necessary, we have

$$\widehat{u}_m \to u^* \quad \text{weakly in } H^1(C)$$

with some $u^* \in H^1(C)$. From Proposition 1 it follows that $u^* = u_0 \circ T_0$ and $u_0 \in S(\Omega_0)$. Hence the pair (Ω_0, u_0) is admissible for (23).

We will prove now that the pair (Ω_0, u_0) is an optimal one for (23). We proceed as in Theorem 3.1 of [16]. Firstly, combining Remark 2(ii) and Remark 3, we deduce that for any compact G in Ω_0, there is an $m_G > 0$ such that $G \subset \Omega_m$ for all $m \geq m_G$. Next, we show that for any such G we have $||u_m - u_0||_{L^2(G)} \to 0$ which implies $u_m \to u_0$ locally. To this end, let $\underline{\widehat{u}_m}$ and $\underline{u^*}$ denote the functions in $L^2(\mathbb{R}^N)$ obtained from $\widehat{u}_m$ and u^*, respectively, by extending them by zero outside

C. From (25) and the compactness of the embedding $H^1(C) \subset L^2(C)$, it follows that

$$\underline{\widehat{u}_m} \to \underline{u^*} \quad \text{in } L^2(\mathbb{R}^N).$$

But this implies that

$$\underline{u_m} \to \underline{u_0} \quad \text{in } L^2(\mathbb{R}^N),$$

where

$$\underline{u_m}(x) = \begin{cases} u_m(x), & \text{if } x \in \Omega_m \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \Omega_m, \end{cases}$$

$$\underline{u_0}(x) = \begin{cases} u_0(x), & \text{if } x \in \Omega_0 \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \Omega_0, \end{cases}$$

which means that $||u_m - u_0||_{L^2(G)} \to 0$, so $u_m \to u_0$ locally in $\mathbb{R}^N$. Hence, due to the hypothesis $H(J)$, we conclude that (Ω_0, u_0) solves the problem (23). □

In the case where the cost criterion is of integral form (24), the sufficient conditions for its lower semicontinuity with respect to the local convergence were given by Serrin in [32], for instance, (in the most simple case) the integrand $L(x, u, p)$ should be nonnegative, continuous in (x, u, p) and strictly convex in p.

In the remaining part of this section we will present a generalization of Theorem 4.1 to the case in which $TK(\Omega)$ varies with T.

DEFINITION 4. Let S_m, S be subsets of a Hausdorff topological space (Z, τ). The sequential Kuratowski lower and upper limits are defined respectively by $\tau - \liminf S_m = \{z \in Z : \exists\, z_m \in S_m, z_m \to z \text{ in } \tau - Z, \text{ as } m \to +\infty\}$ and $\tau - \limsup S_m = \{z \in Z : \exists\,\{m_\nu\}, z_{m_\nu} \in S_{m_\nu}, z_{m_\nu} \to z \text{ in } \tau - Z, \text{as } \nu \to +\infty\}$. When Z is a Banach space, we say that S_m converge to S in the Mosco sense (denoted by $S_m \xrightarrow{M} S$) if and only if $w - \limsup S_m = s - \liminf S_m = S$, where the letters w and s stand, respectively, for the weak and strong topologies on Z.

REMARK 5. It can be easily observed that $S_m \xrightarrow{M} S$ iff $w - \limsup S_m \subseteq S \subseteq s - \liminf S_m$, which in turn, is equivalent to the following two conditions:

(i) if $z_{m_\nu} \in S_{m_\nu}$ and $z_{m_\nu} \to z$ weakly in Z, then $z \in S$;

(ii) for every $z \in S$, there exists $z_m \in S_m$ such that $z_m \to z$ strongly in Z.

The dependence of the sets $TK(\Omega)$ on T is specified in the following hypothesis

$\underline{H(K)_3}$: For $\Omega_m, \Omega \in \mathcal{O}^{k,\infty}$ and $T_m, T \in \mathcal{F}^{k,\infty}$ with $k \geq 2$ such that $\Omega_m = T_m(C)$, $\Omega_0 = T_0(C)$ and $T_m \to T_0$, $T_m^{-1} \to T_0^{-1}$ in $[W^{1,\infty}(\mathbb{R}^N)]^N$, we have $T_m K(\Omega_m) \xrightarrow{M} T_0 K(\Omega_0)$ in $H^1(C)$, as $m \to +\infty$.

LEMMA 3. *Under the hypotheses* (H_0) *with* $\sigma = \infty$, $H(a)$, $H(\beta)$, $H(K)_1$, $H(K)_3$ *and* $H(f)$, *if* $u_m \in S(\Omega_m)$, *then*

$$||u||_{V(\Omega_m)} \leq C\left(1 + meas(\Omega_m) + ||f||_{V'}\right), \tag{26}$$

where a positive constant C *is independent of* m.

Proof. Arguing as in the proof of Lemma 2, we obtain from (6) that (26) holds with $C(v_m)$, where

$$C(v_m) \leq b_1||v_m|| + b_2 \tag{27}$$

for every $v_m \in K(\Omega_m)$ and constants $b_1, b_2 > 0$ independent of m.

Let $v^* \in T_0K(\Omega_0)$. Then, by hypothesis $H(K)_3$, we find $\widehat{v}_m \in T_mK(\Omega_m)$ such that $\widehat{v}_m \to v^*$ in $H^1(C)$. Let $v_m = \widehat{v}_m \circ T_m^{-1}$. We have $v_m \in K(\Omega_m)$ and by the estimates of Remark 4.1 in [20], $||v_m||_{V(\Omega_m)}$ is bounded independently of m. By using the functions v_m in (27), we obtain (26). □

PROPOSITION 2. *If hypotheses* (H_0) *with* $\sigma = \infty$, $H(a)$, $H(\beta)$, $H(K)_1$, $H(K)_3$ *hold and* $f \in L^2(\Omega)$, *then the assertion of Proposition 1 holds.*

Proof. It goes similarly to that of Proposition 1 with some modifications where $K(\Omega)$ is replaced with $TK(\Omega)$ changing with $T \in \mathcal{F}^{k,\infty}$. Let Ω_m, Ω_0, T_m, T_0, $u_m, \widehat{u}_m$, $\chi_m, \widehat{\chi}_m$, $f_m, \widehat{f}_m$ be as before. In the place of (12), we now have: $\widehat{u}_m \in H^1(C)$ are such that

$$a_{T_m}(\widehat{u}_m, \widehat{v}_m - \widehat{u}_m) + (\widehat{\chi}_m, \widehat{v}_m - \widehat{u}_m) \geq (\widehat{f}_m, \widehat{v}_m - \widehat{u}_m), \tag{28}$$

for every $\widehat{v}_m \in T_mK(\Omega_m)$. The relation (13) does not need any modification. Moreover, analogously to (14) and (16), we have

$$\widehat{u}_m \to u^* \quad \text{weakly in } H^1(C) \ \text{ and in } L^2(C), \tag{29}$$

$$\widehat{\chi}_m \to \chi^* \quad \text{weakly in } L^2(C). \tag{30}$$

Since $\widehat{u}_m \in T_mK(\Omega_m)$, from (29) and $H(K)_3$, we get $u^* \in T_0K(\Omega_0)$.

Let $\widehat{v} \in T_0K(\Omega_0)$ be fixed. Then, there is a sequence $\widehat{v}_m \in T_mK(\Omega_m)$ such that $\widehat{v}_m \to \widehat{v}$ in $H^1(C)$. These functions $\widehat{v}_m$ will be used in (28). From the inequality

$$|(\widehat{f}_m, \widehat{v}_m - \widehat{u}_m) - (\widehat{f}_0, \widehat{v} - u^*)| \leq ||\text{det}J_{T_m} - \text{det}J_{T_0}|| \int_C |\widehat{f}_m(\widehat{v}_m - \widehat{u}_m)|\, dX + \\ +||\text{det}J_{T_0}|| \left(|\widehat{f}_m - \widehat{f}_0|\,|\widehat{v}_m - \widehat{u}_m| + |\widehat{f}_0|\,[\,|\widehat{v}_m - \widehat{v}| + |\widehat{u}_m - u^*|\,]\right)$$

and the convergences (29), $\widehat{f}_m \to \widehat{f}_0$, $\widehat{v}_m \to \widehat{v}$ in $L^2(C)$ and $\text{det}J_{T_m} \to \text{det}J_{T_0}$ in $L^\infty(\mathbb{R}^N)$, it follows that

$$(\widehat{f}_m, \widehat{v}_m - \widehat{u}_m) \to (\widehat{f}_0, \widehat{v} - u^*), \quad \text{as } m \to +\infty. \tag{31}$$

Similarly to (18) and (20), we have

$$\begin{cases} (\widehat{\chi}_m, \widehat{v}_m - \widehat{u}_m) \to (\chi^*, \widehat{v} - u^*), \\ a_{T_m}(\widehat{u}_m, \widehat{v}_m - \widehat{u}_m) - a_{T_0}(\widehat{u}_m, \widehat{v}_m - \widehat{u}_m) \to 0. \end{cases} \tag{32}$$

Due to the strong convergence of $\{\widehat{v}_m\}$ in $H^1(C)$, we obtain

$$\lim_m |a_{T_0}(\widehat{u}_m, \widehat{v}_m - \widehat{v})| \le M \lim_m ||\widehat{u}_m||_{H^1(C)} \, ||\widehat{v}_m - \widehat{v}||_{H^1(C)} = 0. \tag{33}$$

From (28), using (29)–(33), we get

$$\begin{aligned} (\widehat{f}_0, \widehat{v} - u^*) \; = \; & \lim_m (\widehat{f}_m, \widehat{v}_m - \widehat{u}_m) \le \liminf_m a_{T_0}(\widehat{u}_m, \widehat{v}_m - \widehat{u}_m) + \\ & + \lim_m \left[a_{T_m}(\widehat{u}_m, \widehat{v}_m - \widehat{u}_m) - a_{T_0}(\widehat{u}_m, \widehat{v}_m - \widehat{u}_m) \right] + \\ & + \lim_m (\widehat{\chi}_m, \widehat{v}_m - \widehat{u}_m) \le \lim_m a_{T_0}(\widehat{u}_m, \widehat{v}_m - \widehat{v}) + \\ & + \limsup_m a_{T_0}(\widehat{u}_m, \widehat{v} - \widehat{u}_m) + (\chi^*, \widehat{v} - u^*) \le \\ \le \; & a_{T_0}(u^*, \widehat{v} - u^*) + (\chi^*, \widehat{v} - u^*). \end{aligned}$$

Since $\widehat{v} \in T_0 K(\Omega_0)$ is arbitrary, by the same reasoning as in the proof of Proposition 1, we have

$$a(u_0, v - u_0) + (\chi_0, v - u_0)_{L^2(\Omega_0)} \ge (f, v - u_0)_{L^2(\Omega_0)}, \quad \forall\, v \in K(\Omega_0),$$

where $u_0 = u^* \circ T_0^{-1}$. Since the relation $\chi_0(x) \in \partial j(u_0(x))$ for a.e. $x \in \Omega_0$ can be proved exactly as in Proposition 1, we deduce that $u_0 \in S(\Omega_0)$ and we are done. □

The proof of the following theorem uses Lemma 3 and Proposition 2. It is completely analogous to the one of Theorem 3 and is therefore omitted.

THEOREM 4. *Under hypotheses* (H_0) *with* $\sigma = \infty$, $H(a)$, $H(\beta)$, $H(K)_1$, $H(K)_3$, $H(J)$ *and* $f \in L^2(\Omega)$, *problem (23) has a solution.*

REMARK 6. Having in mind Remark 4, we can prove Theorem 4 in the case of more general right hand side of (HVI) i.e. when $f \in H^{-1}(\Omega)$ as well as in the case where $V = H_0^1(\Omega)$.

5. Hemivariational Inequality with Nonlinear Law on the Boundary

The aim of this section is to provide an existence result for shape optimization problems for hemivariational inequalities of the type

$$\begin{cases} \text{find } u \in K(\Omega) \text{ such that} \\ a(u, v - u) + \displaystyle\int_\Gamma j^0(u, v - u)\, d\sigma \ge \langle f, v - u \rangle_{V' \times V} \,, \; \forall\, v \in K(\Omega), \end{cases} \tag{34}$$

where $K(\Omega) \subset V(\Omega) = H^1(\Omega)$ and $\Gamma = \partial\Omega$. This shape design problem is investigated in a similar way as the one for the problem (HVI) of Section 3. Therefore, we restrict ourselves to presentation the main steps of the reasoning, indicating only the points where the essential changes in the arguments (in comparison with Sections 3 and 4) are needed.

We admit the following

DEFINITION 5. An element $u \in K(\Omega)$ is a solution to (34) if there is a function $\chi \in L^2(\Gamma)$ such that

$$a(u, v-u) + \int_\Gamma \chi(v-u)\, d\sigma(x) \geq \langle f, v-u\rangle_{V'\times V}\,, \quad \forall\, v \in K(\Omega) \tag{35}$$

and

$$\chi(x) \in \partial j(u(x)) \text{ for a.e. } x \in \Gamma. \tag{36}$$

Let $S_\Gamma(\Omega)$ denote the set of solutions to (34).

THEOREM 5. *Under the hypotheses $H(a)$, $H(\beta)$, $H(K)_1$ and $H(f)$, the problem (34) admits a solution, i.e. $S_\Gamma(\Omega) \neq \emptyset$. Moreover, if $u \in S_\Gamma(\Omega)$, then we have*

$$||u||_V \leq C(v)\left(1 + \mathit{meas}_{N-1}(\Gamma) + ||f||_{V'}\right),$$

where $C(v)$ satisfies $0 < C(v) \leq b_1||v|| + b_2$ and v is an arbitrary element of $K(\Omega)$.

Proof. The existence of solutions to (34) can be established by using the methods of Chapter 3 of [22] (compare also [30]). For the proof of the estimate, we follow the reasoning of the proof of Lemma 2. From (36) and $H(\beta)$, we get

$$\begin{cases} (\chi, u)_{L^2(\Gamma)} \geq -\rho_1\rho_2\, \mathrm{meas}_{N-1}(\Gamma), \\ |\chi| \leq c_2\left(\sqrt{\mathrm{meas}_{N-1}(\Gamma)} + |u|\right), \end{cases} \tag{37}$$

where $c_2 > 0$ and $|\cdot|$ stands for the norm in $L^2(\Gamma)$. Next, by the coercivity of $a(\cdot,\cdot)$, (35) and (37), it follows that

$$\begin{aligned} \alpha||u-v||_V^2 \;\leq\; & \rho_1\rho_2\, \mathrm{meas}_{N-1}(\Gamma) + M||v||\,||v-u|| + ||f||_{V'}||u-v|| + \\ & + c_2\left(\sqrt{\mathrm{meas}_{N-1}(\Gamma)} + |u|\right)|v| \end{aligned}$$

for all $v \in K(\Omega)$. In order to obtain the desired estimate, we proceed subsequently as in the proof of Lemma 2 using additionally the fact that $V(\Omega) \subset L^2(\Gamma)$ continuously. □

We recall the following two results which are needed in the sequel and whose proofs can be found in Lemma 4.7 and Lemma 4.8 of [20], respectively.

LEMMA 4. *(Change of variables in boundary integrals). Let C be an open bounded subset of $\mathbb{R}^N$ with a locally Lipschitz boundary ∂C and let $\Omega = T(C)$ with $T \in \mathcal{F}^{1,\overline{c}}$. If $g \in L^1(\partial\Omega)$, then $g \circ T \in L^1(\partial C)$ and*

$$\int_{\partial\Omega} g d\sigma = \int_{\partial C} (g \circ T) \left|\det J_T\right| \left| J_T^{-t} \nu \right|_{\mathbb{R}^N} d\sigma,$$

where ν is the exterior unit normal to ∂C.

LEMMA 5. *(Transport of the normal through diffeomorphism) Let C be an open subset of $\mathbb{R}^N$ with a locally Lipschitz boundary ∂C, let $\Omega = T(C)$ and $T \in \mathcal{F}^{1,\overline{c}}$. Then,*

$$\nu_\Omega(T(X)) = \frac{J_T^{-t}(X)\nu_C(X)}{\left|J_T^{-t}(X)\nu_C(X)\right|_{\mathbb{R}^N}} \quad a.e.\ X \in \partial C,$$

(where ν_D denotes the exterior unit normal to set D).

REMARK 7. In connection with Lemma 4, it should be stressed here that if T is in $\mathcal{F}^{1,\infty}$ (and not in $\mathcal{F}^{1,\overline{c}}$), then we are not able to define a summable functions on $\partial\Omega$ since $\Omega = T(C)$ does not have, in general, a locally Lipschitz boundary. For this reason we consider in the shape optimization problem for (34) the class of admissible domains which are obtained as images of a set C through mappings T from the space $\mathcal{F}^{k,\overline{c}}$ and not only from $\mathcal{F}^{k,\infty}$ as in Section 3.

In what follows we use a bit stronger, in view of the above remark, hypothesis than $H(K)_2$:

$\underline{H(K)_4}$: $K = TK(\Omega) \subset H^1(C)$ is independent of T for every $T \in \mathcal{F}^{k,\overline{c}}$, with $k \geq 2$.

PROPOSITION 3. *Assume that (H_0) with $\sigma = \overline{c}$, $H(a)$, $H(\beta)$, $H(K)_1$, $H(K)_4$ hold and $f \in L^2(\Omega)$. Then the statement of Proposition 1 remains true provided the set $S(\Omega)$ is replaced by $S_\Gamma(\Omega)$; i.e. the map $\mathcal{B} \ni \Omega \mapsto S_\Gamma(\Omega) \subset V(\Omega)$ has a closed graph in the following sense: if $\Omega_m, \Omega_0 \in \mathcal{B}$, $\Omega_m \to \Omega_0$ in $\mathcal{O}^{k,\overline{c}}$, ($k \geq 1$), $u_m \in S_\Gamma(\Omega_m)$, $\widehat{u}_m = u_m \circ T_m$, $\widehat{u}_m \to u^*$ weakly in $H^1(C)$, then $u^* = u_0 \circ T_0$ for some $u_0 \in S_\Gamma(\Omega_0)$, where $\Omega_m = T_m(C)$ and $\Omega_0 = T_0(C)$.*

Proof. Let Ω_m, Ω_0, T_m, T_0 be as in the proof of Proposition 1, $u_m \in S_\Gamma(\Omega)$ and let $\Gamma_m = \partial\Omega_m$. Thus $u_m \in K(\Omega_m)$ and $\chi_m \in L^2(\Gamma_m)$ are such that

$$a(u_m, v - u_m) + \int_{\Gamma_m} \chi_m(v - u_m)\, d\sigma \geq (f, v - u_m)_{L^2}, \quad \forall\, v \in K(\Omega_m) \tag{38}$$

and

$$\chi_m(x) \in \partial j(u_m(x)) \text{ for a.e. } x \in \Gamma_m. \tag{39}$$

Applying the transformation $x = T_m(X)$ and using Lemma 4, we rewrite (38) and (39) in the following equivalent form

$$a_{T_m}(\widehat{u}_m, v - \widehat{u}_m) + (\widehat{\chi}_m, v - \widehat{u}_m) \geq (\widehat{f}_m, v - \widehat{u}_m), \quad \forall\, v \in K \subset H^1(C) \tag{40}$$

and

$$\widehat{\chi}_m(X) \in \partial j(\widehat{u}_m(X)) \text{ for a.e. } X \in \partial C, \tag{41}$$

where $\widehat{u}_m = u_m \circ T_m$, $\widehat{\chi}_m = \chi_m \circ T_m$, $\widehat{f}_m = f \circ T_m$,

$$(\widehat{\chi}_m, \widehat{u}_m) = \int_{\partial C} \widehat{\chi}_m\, \widehat{u}_m\; |\mathrm{det} J_{T_m}|\; \left|J_{T_m}^{-t} \nu\right|_{\mathbb{R}^N} d\sigma \tag{42}$$

and the expressions for $a_{T_m}(\cdot,\cdot)$ and $(\widehat{f}_m, v)$ are the same as in the proof of Proposition 1. We keep v in (40) to be fixed in K since K satisfies $H(K)_4$. As before, we have $\widehat{u}_m \to u^*$ weakly in $H^1(C)$ and strongly in $L^2(C)$ with $u^* \in K$. Moreover, since $H^1(C) \subset L^2(\partial C)$ compactly, it follows that

$$\widehat{u}_m \to u^* \quad \text{in } L^2(\partial C). \tag{43}$$

Subsequently, from (41) and $H(\beta)(iii)$, we obtain

$$|\widehat{\chi}_m| \leq c_4(\mathrm{meas}_{N-1}(\partial C) + |\widehat{u}_m|) \tag{44}$$

with $c_4 > 0$ ($|\cdot|$ being the $L^2(\partial C)$ norm). From (43) and (44), after passing to a subsequence if necessary, we have

$$\widehat{\chi}_m \to \chi^* \quad \text{weakly in } L^2(\partial C) \tag{45}$$

with some $\chi^* \in L^2(\partial C)$.

We claim that

$$(\widehat{\chi}_m, \widehat{u}_m) \to \int_{\Gamma_0} \chi_0 u_0\, d\sigma, \tag{46}$$

where $\Gamma_0 = \partial\Omega_0$, $\chi_0 = \chi^* \circ T_0^{-1}$ and $u_0 = u^* \circ T_0^{-1}$. Indeed, taking into account (43), (45) and the convergences $\mathrm{det} J_{T_m} \to \mathrm{det} J_{T_0}$ in $L^\infty(\mathbb{R}^N)$, $J_{T_m}^{-1} \to J_{T_0}^{-1}$ in $L^\infty(\mathbb{R}^N, \mathbb{R}^{2N})$, we can pass to the limit in (42) and we get

$$\lim_m (\widehat{\chi}_m, \widehat{u}_m) = \int_{\partial C} \chi^*(X)\, u^*(X)\; |\mathrm{det} J_{T_0}(X)|\; \left|J_{T_0}^{-t}(X) \nu_C(X)\right|_{\mathbb{R}^N} d\sigma(X).$$

Next, applying the transformation $X = T_0^{-1}(x)$ and Lemma 4 again, we have

$$\begin{aligned}\lim_m (\widehat{\chi}_m, \widehat{u}_m) = \int_{\Gamma_0} &\chi_0(x)\, u_0(x)\; \left|\mathrm{det} J_{T_0}(T_0^{-1}(x))\right| \times \\ &\times \left|J_{T_0}^{-t}(T_0^{-1}(x)) \nu_C(T_0^{-1}(x))\right|_{\mathbb{R}^N} \left|\mathrm{det} J_{T_0^{-1}}(x)\right| \left|J_{T_0^{-1}}^{-t}(x) \nu_{\Omega_0}(x)\right|_{\mathbb{R}^N} d\sigma(x).\end{aligned}$$

Using the relations $J_{T_0}^{-1}(T_0^{-1}(x)) = J_{T_0^{-1}}(x)$, $\det J_{T_0}(T_0^{-1}(x)) \cdot \det J_{T_0^{-1}}(x) = 1$ a.e on $\mathbb{R}^N$ and Lemma 5, we obtain

$$\lim_m (\widehat{\chi}_m, \widehat{u}_m) = \int_{\Gamma_0} \chi_0(x)\, u_0(x) \left| J_{T_0}^{-t}(T_0^{-1}(x)) \nu_C(T_0^{-1}(x)) \right|_{\mathbb{R}^N} \times$$
$$\times \left| J_{T_0}^{t}(T_0^{-1}(x)) \frac{J_{T_0}^{-t}(T_0^{-1}(x)) \nu_C(T_0^{-1}(x))}{\left| J_{T_0}^{-t}(T_0^{-1}(x)) \nu_C(T_0^{-1}(x)) \right|_{\mathbb{R}^N}} \right|_{\mathbb{R}^N} d\sigma(x).$$

which proves the claim.

Analogously as in the proof of Proposition 1, we get (17) and (20), which together with (46) allow to pass to the limit in (40). Moreover, also similarily as before, we use the convergence theorem (cf. [1]) to deduce that $\chi^*(X) \in \partial j(u^*(X))$ for a.e. $X \in \partial C$. Thus, we have

$$\begin{cases} a(u_0, v - u_0) + (\chi_0, v - u_0)_{L^2(\Gamma_0)} \geq (f, v - u_0)_{L^2(\Omega_0)}, \quad \forall\, v \in K \\ \chi_0(x) \in \partial j(u_0(x)) \text{ for a.e. } x \in \Gamma_0. \end{cases}$$

Hence we conclude immediately that $u_0 \in S_\Gamma(\Omega_0)$ and $u^* = u_0 \circ T_0$, which completes the proof. □

The optimal shape design problem for hemivariational inequality of type (34) reads as follows

$$\begin{cases} \text{find } (\Omega^*, u^*) \in \bigcup_{\Omega \in \mathcal{B}} (\Omega \times S_\Gamma(\Omega)) \text{ such that} \\ \mathcal{J}(\Omega^*, u^*) = \min_{\Omega \in \mathcal{B}} \min_{v \in S_\Gamma(\Omega)} \mathcal{J}(\Omega, v), \end{cases} \tag{47}$$

where $\mathcal{B} \subset \mathcal{O}^{k,\overline{c}}$ ($k \geq 2$) denotes a family of admissible shapes, $S_\Gamma(\Omega)$ is the solution set of (34) and the functional J is of the form (24).

The existence of optimal shapes in the above problem follows from similar arguments as given in Theorem 3.

THEOREM 6. *Under hypotheses* (H_0) *with* $\sigma = \overline{c}$, $H(a)$, $H(\beta)$, $H(K)_1$, $H(K)_4$, $H(J)$ *and* $f \in L^2(\Omega)$, *the problem (47) admits a solution.*

When the sets $TK(\Omega)$ vary with T, we need the following hypothesis

$\underline{H(K)_5}$: For $\Omega_m, \Omega \in \mathcal{O}^{k,\overline{c}}$ and $T_m, T \in \mathcal{F}^{k,\overline{c}}$ with $k \geq 2$ such that $\Omega_m = T_m(C)$, $\Omega_0 = T_0(C)$ and $T_m \to T_0$, $T_m^{-1} \to T_0^{-1}$ in $[W^{1,\overline{c}}(\mathbb{R}^N)]^N$, we have $T_m K(\Omega_m) \xrightarrow{M} T_0 K(\Omega_0)$ in $H^1(C)$, as $m \to +\infty$.

In this case the following theorem can be established.

THEOREM 7. *If hypotheses* (H_0) *with* $\sigma = \overline{c}$, $H(a)$, $H(\beta)$, $H(K)_1$, $H(K)_5$, $H(J)$ *hold and* $f \in L^2(\Omega)$, *then problem (47) has a solution.*

6. Comments on Applications

The mathematical results for HVIs have had a significant impact on several areas of mechanics. We give three examples coming from mechanics which well fit the framework of domain optimization problems outlined in this paper.

1) Plane elasticity problem (see [18]). A contact problem of an elastic body with a rubber layer situated on a rigid foundation. There is a nonmonotone law between displacement and boundary force which leads to problem of type (2).

2) Skin effects in elasticity (see [27], [28]). This is a class of problems in plane elasticity theory where the adhesive or frictional effects take place in a set $\Omega' \subseteq \Omega$. There is a multivalued (one dimensional) law between the displacement and the reaction of the constraint introducing the skin effect; it leads to problem of type (1).

3) Semipermeable media (see [24] and [22], p. 186). There are two classes of semipermeability problems, the interior and the boundary ones, which lead to (1) and (2), respectively.

For other examples to which our results can be applied, we refer to [24], [27] and [22].

REMARK 8. The typical cost functionals of the form (24) arising in structural mechanics, electricity, fluid flow etc. are following: $L(x,u,p) = \rho(x)$, $L(x,u,p) = |u - u_0|^2$, $L(x,u,p) = |p|^2$ which correspond, respectively, to minimization of weight (ρ being a density function), minimization of displacement (or of the deviation from the desired state u_0), and minimization of stresses.

REMARK 9. There is a vast literature concerning the convergence of Mosco for unilateral nonempty convex sets of the type

$$K(g) = \{v \in W_0^{1,p}(\Omega) \,:\, v \geq g \text{ a.e. in } \Omega\}$$

where $1 < p < \infty$. It is known (cf. [2]), for instance, that $K(g_n) \xrightarrow{M} K(g)$, as $n \to \infty$, if one of the following conditions holds

$$g_n \to g \;\text{ in }\; W^{1,p}(\Omega),$$

$$g_n \to g \;\text{ weakly in }\; W^{1,p+\varepsilon}(\Omega),\; \text{for some } \varepsilon > 0.$$

A necessary and sufficient condition for the Mosco convergence of $K(g_n)$, expressed in terms of the convergence of the $W_0^{1,p}(\Omega)$–capacity of the level sets $\{x \in \Omega \,:\, g_n(x) > t\}$ has been established in [7].

REMARK 10. For the convenience of the reader, we present, following [16] some examples of unilateral sets which are met in applications:

$$\begin{aligned} K_1(\Omega) &= \{v \in H^1(\Omega) \,:\, v = b \text{ on } \partial\Omega,\; v \geq c \text{ a.e. in } \Omega\}, \\ K_2(\Omega) &= \{v \in H^1(\Omega) \,:\, v \geq d \text{ a.e. in } \Omega\}, \\ K_3(\Omega) &= \{v \in H_0^1(\Omega) \,:\, v(x) \geq \varphi(x) \text{ a.e. in } \Omega\}, \end{aligned}$$

where b, c, d are constants, $\varphi \in W^{1,p}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and $p > 2$. It can be shown that the sets $K_1(\Omega)$ and $K_2(\Omega)$ satisfy hypothesis $H(K)_2$ while $K_3(\Omega)$ satisfies $H(K)_3$.

REMARK 11. For the cost functionals of the form $\mathcal{J}(\Omega, u) = \int_\Omega L(x, u(x))\,dx$, we can obtain the existence of solutions to the problems (23) and (47) without referring to the local convergence of functions. In this case, we suppose that $L\colon \mathbb{R}^N \times \mathbb{R} \to [0, +\infty)$ is a Borel function, $L(x, \cdot)$ is l.s.c. and $L(x, v) \leq c\,(1 + |v|^2)$ for all $x \in \mathbb{R}^N$, $v \in \mathbb{R}$ with $c > 0$. For details, we refer to Theorem 3 in [10].

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